MOMENTS OF AIRY FUNCTIONS AS ULTERIOR MOTIVES

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2021 TMS ANNUAL MEETING

A. Rational Hodge structures over $\mathbb{Q}$

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$V_B = H^r_{sing}(X(\mathbb{C}), \mathbb{Q})$

$V_{dR} = H^r_{dR}(X/\mathbb{Q})$

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V_B : $\mathbb{Q}$-v.s.

(V_{dR}, F^p) : filtered $\mathbb{Q}$-v.s.

comp : $V_{dR} \otimes \mathbb{C} \xrightarrow{\sim} V_B \otimes \mathbb{C}$ comparison isomorphism satisfying

- Hodge decomposition

$$V_B \otimes \mathbb{C} = \bigoplus_{p+q=r} V^{p,q}$$

$$V^{p,q} = \mathrm{comp}\left(F^p V_{dR} \otimes \mathbb{C}\right) \cap \overline{\mathrm{comp}\left(F^q V_{dR} \otimes \mathbb{C}\right)}$$

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- Compatibility of complex conjugation

$$\begin{array}{ccccccc} V_{dR} & \otimes & \mathbb{C} & \simeq & V_B & \otimes & \mathbb{C} \\ & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ & & \rho & & \rho_B & \otimes & \rho \end{array}$$

Example: Tate twist of weight 2.

$$H^2(\mathbb{P}^1) \simeq H^1(\mathbb{G}_m = \operatorname{Spec} \mathbb{Q}[z, z^{-1}]) = \mathbb{Q}(-1)$$

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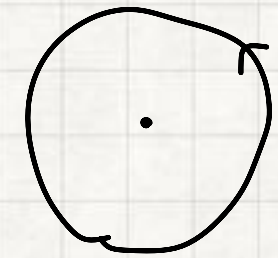
$V_B = \mathbb{Q} \cdot \gamma^\vee$ where $\gamma =$ 

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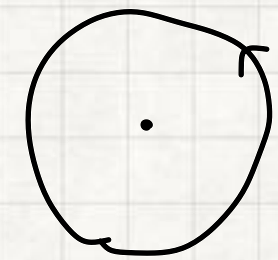
$V_B = \mathbb{Q} \cdot \gamma^\vee$ where $\gamma =$ 

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$$\mathrm{comp}\left(\frac{dz}{z}\right) = \left\langle \gamma^\vee, \frac{dz}{z} \right\rangle = \int_\gamma \frac{dz}{z} \cdot \gamma = 2\pi i \cdot \gamma$$

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- $F^p W_{dR}$ is indexed by $p \in \mathbb{Q}$
- $$W_B \otimes \mathbb{C} = \bigoplus_{\substack{p+q=r \\ p,q \in \mathbb{Q}}} W^{p,q}$$

- Ulterior motive $\mathbb{E}_m$

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$$(\mathbb{E}_m)_B = \{ f : \{0, \dots, m-1\} \to \mathbb{Q} \mid \sum f(i) = 0 \}$$

$$(\mathbb{E}_m)_{dR} = \sum_{a=1}^{m-1} \mathbb{Q} \cdot \Gamma_a \, , \qquad \Gamma_a(i) = \zeta_m^{-ai} \Gamma\left(1 - \frac{a}{m}\right)$$

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Thm (Anderson)

$$\left(\mathbb{E}_m^{\otimes n}\right)_{cl} \simeq H^{n-2}_{prim}(X_m^n)(-1)$$

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So periods of X^n_m contains $(2\pi i)^{-1} \prod \zeta_m^{-a_i b_j} \Gamma\left(1 - \frac{a_i}{m}\right)$

$$\sum_{i=1}^{m-1} a_i = n = \sum_{j=1}^{m-1} b_j$$

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$$V_{dR} = H^r_{dR}(X, f) = \mathbb{H}^r(X, \mathcal{O}_X \xrightarrow{d+df} \Omega^1_X \longrightarrow \cdots)$$

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Deligne, Esnault-Sabbah-Y., Katzarkov-Kontsevich-Pantev, Mochizuki,

There is the irregular Hodge filtration on V_{dR}.

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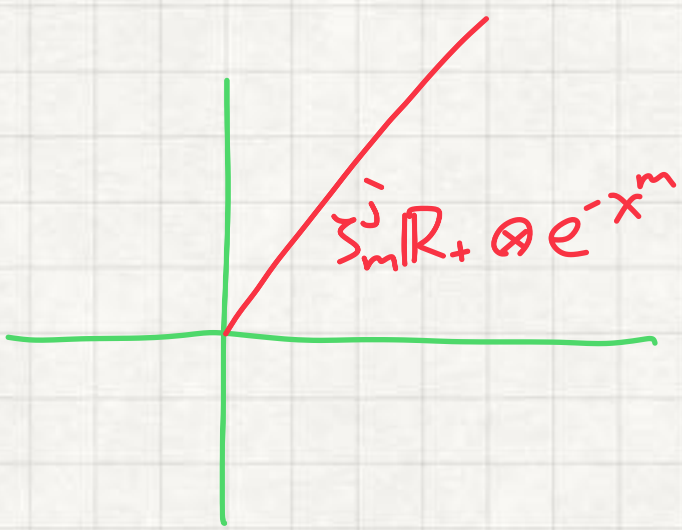
$$V_B^\vee = H_1^{rd}(\mathbb{A}^1, x^m)$$



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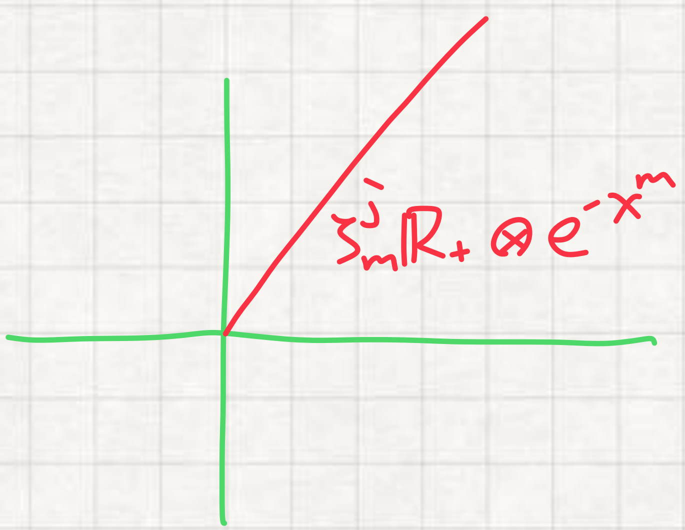

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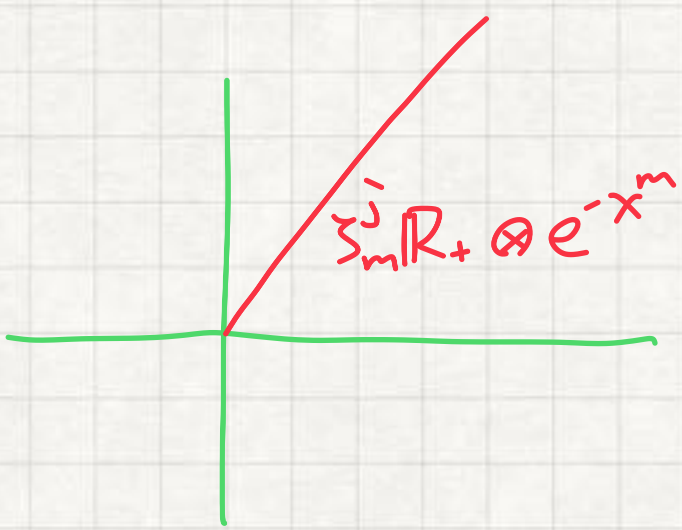

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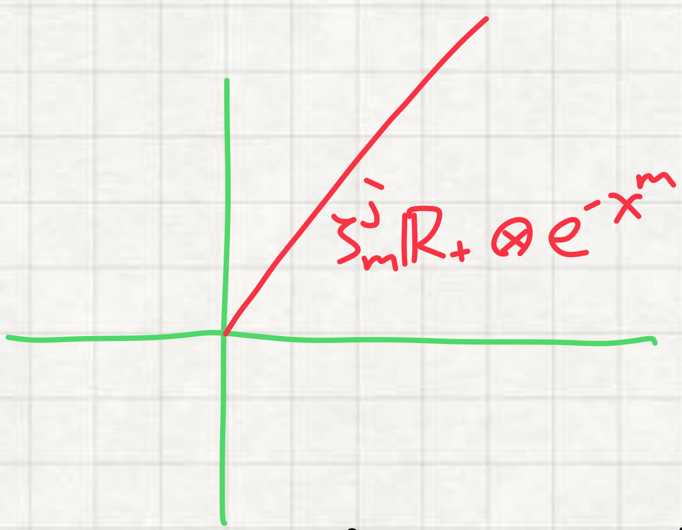

with $\sum_{i=0}^{m-1} \alpha_i = 0$

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One identifies this with E_m via

$$\sum_{i=0}^{m-1} f(i) = 0 \quad \leftrightarrow \quad \sum f(i) \cdot \zeta_m^i \mathbb{R}_+ \otimes e^{-x^m}$$

$$\Gamma_a \quad \leftrightarrow \quad v_{m-a}$$

Anderson's result.

$$\mathbb{E}_m^{\otimes n} = H^n(\mathbb{A}^n, -\sum_{i=1}^n x_i^m)$$

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$$0 \to H^{k+1}\Big(\mathbb{A}^{n+1}, -\sum_{i=0}^{n} x_i^m\Big) \to H^{n+1}\Big(\mathbb{G}_m \times \mathbb{A}^n, -\sum_{i=0}^{n} x_i^m\Big) \to H^n\Big(\mathbb{A}^n, -\sum_{i=1}^{n} x_i^m\Big)(-1) \to 0$$

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$$\Big\| \quad \begin{array}{l} x_0 = t \\ x_i = t y_i \end{array}$$

$$H^{n+1}\left(\mathbb{G}_m \times \mathbb{A}^n, -t^m\left(1+\sum_{i=1}^{n} y_i^m\right)\right)$$

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Finally $\quad \mathrm{gr}^W_n H^{n-1}\left(\left(1+\sum_{i=1}^{n} y_i^m\right)\right) \xrightarrow{\sim} H^{n-2}_{\mathrm{prim}}(X^n_m)(-1) \qquad \square$

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non-classical part: $\quad \left(E_3 \otimes \text{classical}\right)^{\mu_3}$

classical $=$ a sub of $\quad H^k(U)$,

$$U = \left(t^3 - \sum \tfrac{1}{3}x_i^3 - x_i\right) \subset \mathbb{A}^1_s \times (\mathbb{A}^k \setminus Z).$$

THANK YOU